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Drag Reduction of a Non-Newtonian Fluid by Fluid Injection at the Wall

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The flat plate boundary-layer equations for a power law model of a non-Newtonian fluid are formulated in a manner to allow similarity solutions with mass injection at the boundary. The variation of injection velocity with longitudinal position along the plate which allows a similarity solution is determined as a function of the power law exponent N. In the analysis, it is shown that the two-point boundary value problem can be reduced to an equivalent initial value problem. Numerical results are presented for velocity profiles, skin-friction coefficient, displacement and momentum thicknesses for a range of values of the power law exponent, and the dimensionless mass injection parameter.

Introduction

A CONSIDERABLE interest in the flow of fluids that are non-Newtonian has evolved in the past few years. In addition to the technological importance of non-Newtonian fluids, the possibility of reducing skin friction by injection of a non-Newtonian fluid into a Newtonian liquid flow has been demonstrated. Such an application is of importance to hydronautic systems.

Schowalter[1] and Acrivos, Shah, and Peterson[2] were the first investigators to study the external flow over a body of a non-Newtonian fluid. Schowalter formulated the boundary-layer equations in a form amenable to obtaining similarity solutions. More recently, Wells[3] and Lee and Ames[4] have done a detailed analysis of the types of non-Newtonian flows which allow similarity solutions. Acrivos, Shah, and Peterson obtained a similarity solution to the boundary-layer equations for a power law, non-Newtonian fluid flowing along a flat plate at zero angle of attack. Their results appear to be the first published results for the external flow of a non-Newtonian fluid. The Rayleigh problem for a power law fluid was solved by Wells[5] and was studied experimentally by Hermes and Fredrickson[6] for a viscoelastic fluid. A good general reference work on non-Newtonian flows for both internal and external flows is the book by Skelland,[9] although no treatment of boundary-layer flows with mass injection is included.

In the present paper, a solution is obtained of the boundary-layer equations for the laminar flow of power law model non-Newtonian fluid over a flat plate at zero angle of attack with mass injection at the surface. The non-Newtonian fluid is assumed to be incompressible.

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The Analysis

The Ostwald-de Waele shear model (power law model) has been widely used as an acceptable representation of many real fluids with non-Newtonian properties.[3] This model will be used in the present analysis along with the boundary-layer equations.

The two-dimensional, flat plate, boundary-layer equations are

$$(\partial u/\partial x) + (\partial v/\partial y) = 0 \tag{1}$$

$$\rho[u(\partial u/\partial x) + v(\partial u/\partial y)] = (\partial/\partial y)(\tau_{xy}) \tag{2}$$

The shear stress for the Ostwald-de Waele model has the form

$$\tau_{xy} = K\left|\frac{\partial u}{\partial y}\right|^{N-1}\frac{\partial u}{\partial y} \tag{3}$$

where K and N are in general empirically determined constants. For boundary-layer flow in which velocity overshoot is not possible, as in the present case, the velocity gradient is always positive so that Eq. (3) can be written as

$$\tau_{xy} = K(\partial u/\partial y)^N \tag{4}$$

Equations (1, 2, and 4) are to be solved subject to the boundary conditions

$$u(x,0) = 0 \qquad v(x,0) = v_s(x)$$
$$\lim_{y\to\infty} u(x,y) = U_\infty \tag{5}$$

The equations are nondimensionalized as follows:

$$\bar{u} = u/U_\infty \qquad \bar{v} = (v/U_\infty)(Re_L)^{1/(1+N)}$$
$$\bar{x} = x/L \qquad \bar{y} = (y/L)(Re_L)^{1/(1+N)} \tag{6}$$
$$Re_L = \rho U_\infty{}^{(2-n)}L^N/K$$

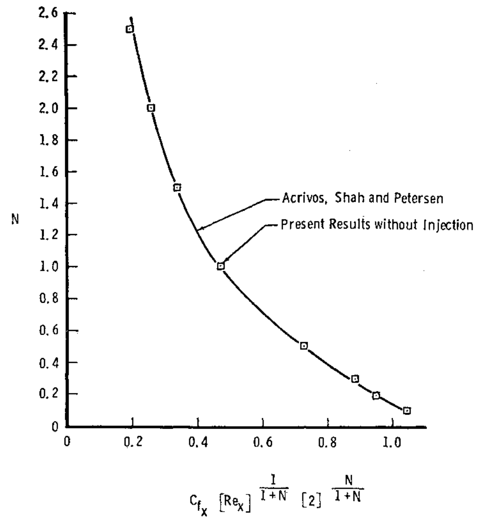


Fig. 1 Comparison of skin-friction coefficient parameter with Ref. 1 results for zero mass injection.

Introducing the similarity variables,

$$\eta = \bar{y}/(2\bar{x})^{1/1+N} \qquad \bar{u} = f'(\eta) \tag{7}$$

transforms the boundary-layer equations into an ordinary differential equation of the Blasius type,

$$f'''(\eta) + [2/N(N+1)]f(\eta)[f''(\eta)]^{2-N} = 0 \tag{8}$$

subject to the boundary conditions

$$f'(0) = 0 \qquad f(0) = \xi \qquad \lim_{\eta\to\infty} f'(\eta) = 1 \tag{9}$$

where

$$\xi = [(1+N)/(2)^{1/(1+N)}](\bar{x})^{N/(1+N)}(Re_L)^{1/(1+N)}[-(v_s/U_\infty)] \tag{10}$$

is the nondimensional blowing parameter.

It is clear that Eq. (8) reduces to the Blasius equation for $N = 1$ and $K = \mu$. From Eq. (10), it follows that a similarity solution with mass injection is possible only if the injection velocity has an x variation of the form $v_s \sim x^{-[N/(1+N)]}$. For $N = 1$, this reduces to the well-known relation $v_s \sim x^{-1/2}$ variation of injection velocity for a similarity solution of the boundary-layer equations for a Newtonian fluid.

Expressions for the skin-friction coefficient C_f, the displacement thickness δ^*, and the momentum thickness θ, may be written in terms of f as

$$C_{fx} = \frac{(\tau_{xy})_{y=0}}{\rho U_\infty^2} = \frac{1}{(2)^{N/(N+1)}(Re_x)^{1/(N+1)}}[f''(0)]^N \tag{11}$$

$$\frac{\delta^*}{x} = \left(\frac{2}{Re_x}\right)^{1/(1+N)} \int_0^\infty (1-f')d\eta \tag{12}$$

$$\frac{\theta}{x} = \left(\frac{2}{Re_x}\right)^{1/(1+N)} \int_0^\infty f'(1-f')d\eta \tag{13}$$

where $Re_x = \rho U_\infty^{(2-N)} x^N/K$.

Method of Solution

Equation (8) with associated boundary conditions is a two-point boundary value problem. It can be reduced to the equivalent problem of solving two third-order equations as initial value problem using the method of Klamkin.[7] A brief discussion of the method follows.

Define a function F by the relation

$$f(\eta) = \lambda^a F(\lambda^{1/3}\eta) \tag{14}$$

where a and λ are constants to be specified. Substituting Eq. (14) into Eq. (8) gives

$$F''' + \{[2/N(N+1)]\lambda^{(2-N)(a+2/3)-1}\} \times [F(F'')^{2-N}] = 0 \tag{15}$$

The exponent on λ in Eq. (15) can be made to vanish by choosing a to have the value

$$a = -[(1-2N)/3(2-N)] \qquad N \neq 2$$

thus reducing Eq. (15) to the form

$$F''' + [2/N(N+1)]F(F'')^{2-N} = 0 \tag{16}$$

Applying the boundary conditions on f from Eq. (9) to Eq. (14) gives

$$F(0) = \xi/\lambda^a \qquad F'(0) = 0$$
$$\lambda = \left[1/\lim_{z\to\infty} F'(z)\right]^{1/(a+1/3)} \tag{17}$$

where $z = \lambda^{1/3}\eta$. Since F is described by a third-order equation, one additional boundary condition may be chosen. Taking the third condition as $F''(0) = 1$ then gives

$$f''(0) = \lambda^{a+2/3} \tag{18}$$

Thus all three conditions on f are now given as initial conditions.

The value of λ is not known a priori as shown by Eq. (17). Thus, the value of the blowing parameter, ξ, cannot be specified a priori in solving the F equation. The procedure followed in the numerical integration is to take $F''(0) = 1$, $F'(0) = 0$, and $F(0) = A$, an assumed constant. With the assumed value of A as initial condition, Eq. (16) is integrated

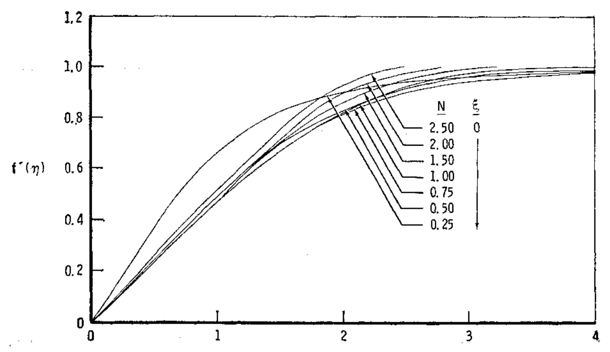


Fig. 2 Velocity profiles with zero mass injection.

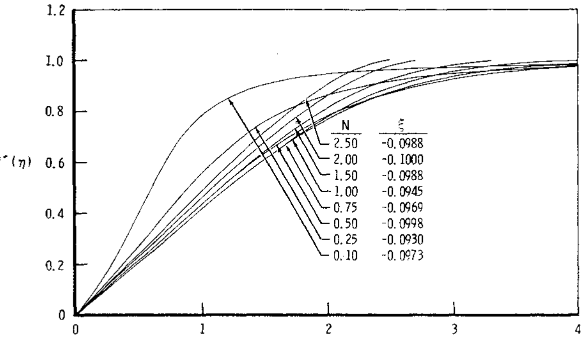
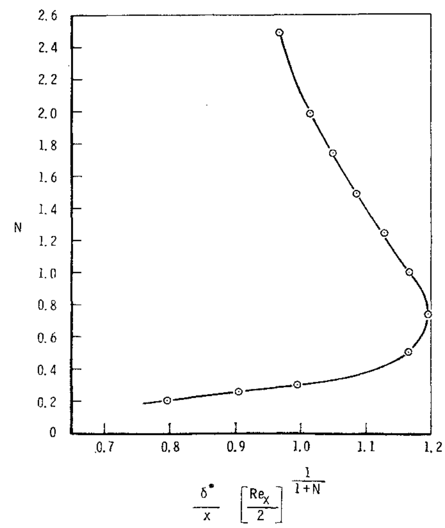
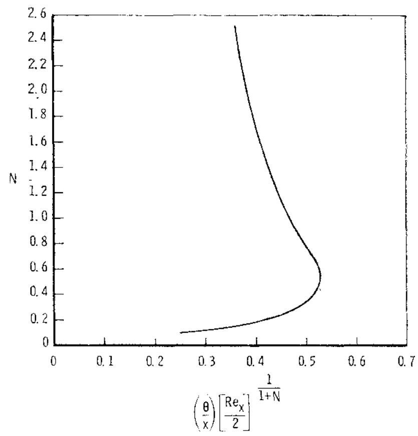


Fig. 3 Velocity profiles for mass injection parameter approximately $|\xi| = 0.1$.

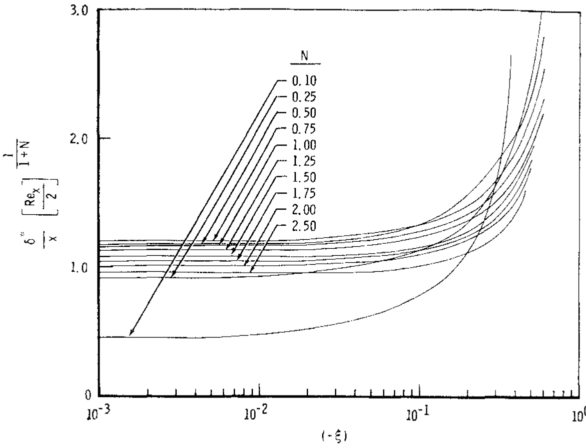


a) **Displacement**

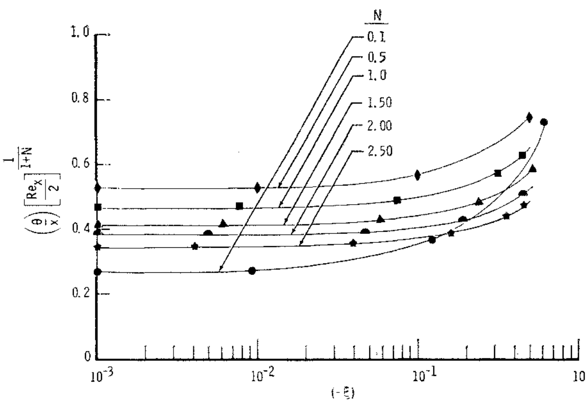


b) **Momentum**

Fig. 4 Thickness parameter for zero mass injection.



a) **Displacement**



b) **Momentum**

Fig. 5 Variation of thickness parameter with mass injection.

numerically as an initial value problem and λ then determined from Eq. (17). The corresponding value of the blowing parameter is determined from the relation

$$\xi = A\lambda^a = \lambda^a F(0) \tag{19}$$

Numerical results were obtained for values of N ranging from 0.1 to 2.5 with ξ ranging from -10^{-4} to -1.

Numerical Results

The only results available in the literature with which the results of the present analysis may be compared are for zero mass injection at the wall. Figure 1 shows the skin-friction coefficient parameter for zero mass injection obtained from the present analysis and the analysis of Acrivos, Shah, Peterson.[2] The agreement between the two results is excellent.

Figure 2 shows typical velocity profiles with zero mass injection for several values of N and Fig. 3 shows the profiles for a mass injection parameter approximately equal to $|\xi| = 0.1$. As one would anticipate, the velocity profiles are made fuller by mass injection at the boundary.

Figures 4a and 4b show the influence of the exponent N on displacement and momentum thickness parameters, respectively, with zero mass injection. The thickness parameters are not monotonic functions of N but exhibit a maximum. The maximum value of the displacement thickness parameter occurs for N approximately 0.75 and the maximum value of the momentum thickness parameter occurs for N approximately 0.54. The absence of monotonicity of boundary-layer thickness with N is also demonstrated in the crossing of the velocity profiles near the outer edge, i.e., as $f'(\eta)$ approaches unity.

Figures 5a and 5b show the influence of mass injection on displacement thickness parameter and momentum thickness parameter for N values in the range $0.1 \leq N \leq 2.5$. For small values of the mass injection parameter, both parameters first increase with increasing N and then decrease with further increases in N, a behavior qualitatively the same as for zero mass injection. For sufficiently large values of the mass injection parameter, a monotonic behavior develops in which both parameters increase with decreasing N for a fixed value of ξ. The monotonic behavior appears to be established for $|\xi| > 0.5$ for the displacement thickness parameter and for $|\xi| > 0.8$ for the momentum thickness parameter.

Figure 6 shows the influence of mass injection on the skin-friction coefficient parameter for N values in the range $0.1 \leq N \leq 2.5$. For the entire range of values of mass injection parameter, the skin-friction coefficient parameter decreases as N increases. For $|\xi| < 0.01$, the reduction is slight, less than about 7%, for all values of N considered. For a given value of ξ, the percentage reduction in skin-friction coefficient parameter is greater for smaller N. For example, for $|\xi| = 0.1$, the skin-friction coefficient parameter reduction below the value for $\xi = 0$ is 35% for $N = 0.1$ and 15% for $N = 2.5$. Extrapolation of the curve for $N = 1$ shows that C_{fx} vanishes for $|\xi| = 0.86$. This agrees favorably with the value $|\xi| = 0.875$ reported in Ref. 8 for the value of mass injection parameter at which the boundary layer is blown off the surface for a Newtonian fluid.

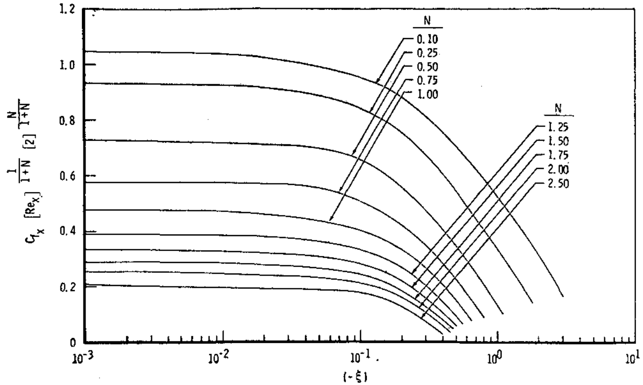


Fig. 6 Variation of skin-friction coefficient parameter with mass injection.

Conclusions

1) A similarity solution of the flat plate boundary-layer equations for a power law model of a non-Newtonian fluid with mass injection at the wall is possible only if the injection velocity has an x variation of the form $v_s \sim x^{-[N/(1+N)]}$.

2) It is shown that the two-point boundary value problem for the similarity solution with mass injection can be reduced to an equivalent initial value problem.

3) Numerical results are presented for the velocity profiles, skin-friction coefficient, displacement thickness parameter, and momentum thickness parameter for N values in the range $0.1 \leq N \leq 2.5$ and mass injection parameter values in the range $10^{-4} \leq |\xi| \leq 1$.

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